Introduction to Laplace Transforms

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Outline

- Review last class
- Why Laplace transforms
  - Definitions of Laplace transforms
  - Getting a transform by integration
  - Finding transforms (and inverse transforms) from tables and theorems
  - Applications to differential equations
- Examples for homogenous and nonhomogeneous equations

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Review Last Class

- Bessel's Equation $\frac{d^2y(x)}{dx^2} + \frac{1}{x}\frac{dy(x)}{dx} + \frac{x^2 - \nu^2}{x^2}y = 0$
- Solution for non-integer ν is y = $AJ_\nu(x)$ + $BJ_{-\nu}(x)$
- Solution for integer ν = n is y = $AJ_n(x)$ + $BY_n(x)$
- Can find Bessel functions in tables and from Excel (integer only) or MATLAB (both integer and non-integer)

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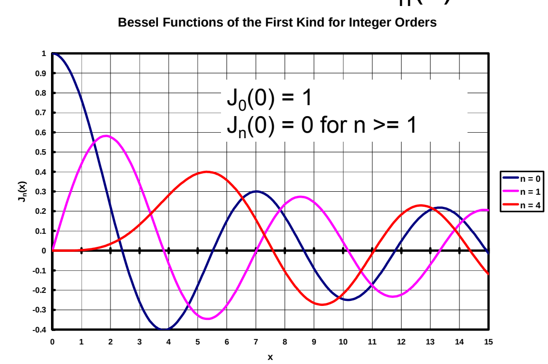
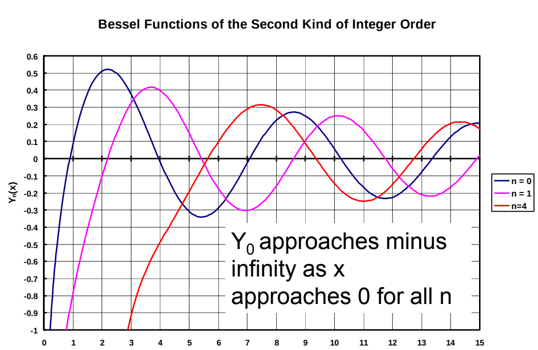
Bessel Functions $J_n(x)$



Bessel Functions $Y_n(x)$



Laplace Transform Definition

- Transforms from a function of time, f(t), to a function in a complex space, F(s), where s is a complex variable
- The transform of a function, is written as F(s) = $\mathcal{L}$[f(t)] where $\mathcal{L}$ denotes the Laplace transform
- Laplace transform defined as the following integral: $\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)dt = F(s)$
- Have tables of F(s) ⇔ f(t)

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Simple Laplace Transforms

f(t)	F(s)	f(t)	F(s)
t^n	$n!/s^{n+1}$	$e^{at}\sin \omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$
t^x	$\Gamma(x+1)/s^{x+1}$		
e^{at}	$1/(s-a)$	$e^{at}\cos \omega t$	$\frac{(s-a)}{(s-a)^2+\omega^2}$
$\sin \omega t$	$\omega/(s^2+\omega^2)$		
$\cos \omega t$	$s/(s^2+\omega^2)$	Additional transforms in pp 264-267/248-251 of Kreyszig 9th/10th edition	
$\sinh \omega t$	$\omega/(s^2-\omega^2)$		
$\cosh \omega t$	$s/(s^2-\omega^2)$		

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Finding Laplace Transforms

- Although we can use the transform definition to find transforms, we usually rely on tables
- Simple integration: find $\mathscr{L}[e^{at}]$
- $\mathscr{L}[e^{at}] = \int_0^\infty e^{-st}e^{at}dt = \frac{1}{a-s}e^{(a-s)t}\Big]_0^\infty = \frac{1}{s-a}$
- Such integration is for example only; we can find such simple transforms in the transform tables

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Why Laplace Transforms?

- Simplified technique for solving initial value problems (IVP)
- Use transform tables to get transforms labeled F(s) for usual functions, f(t)
  - Transforms differential equations (with t as independent variable) into algebraic equations (with s as algebraic variable) and initial condition(s)
  - Tables used to transform equations terms from f(t) to F(s) and vice versa

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Why Laplace Transforms (cont'd)

- Transformed ODE in s space is manipulated to get only F(s) forms available in transform table
- Use transform tables to get transforms from terms in F(s) back to f(t) term
- You now have the solution!
- Simple example: find y(t) for dy/dt + ky = 0 with y(0) = a

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Simple Example

- Find y(t) for dy/dt + ky = 0
- From transform table $\mathscr{L}$(dy/dt) = sY(s) – y(0) and $\mathscr{L}$[ky(t)] = kY(s)
- Transformed equation of dy/dt + ky = 0 is sY(s) – y(0) + kY(s) = 0
- Find Y(s) by algebra: Y(s) = y(0)/(s – k)
- From transform table for Y(s) = 1/(s – a), y(t) = e^{-at}, so solution is y(t) = y(0)e^{-kt}

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Basic Transform Properties

- Linearity Theorem
- $\mathscr{L}[af_1(t) + bf_2(t)] = a\mathscr{L}[f_1(t)] + b\mathscr{L}[f_2(t)]$
- First shifting theorem
- If $\mathscr{L}[f(t)] = F(s)$ then $\mathscr{L}[e^{at}f(t)] = F(s-a)$
- Derivative transforms for $\mathscr{L}[f(t)] = F(s)$
  - $\mathscr{L}[df/dt] = sF(s) - f(0)$
  - $\mathscr{L}[d^2f/dt^2] = s^2F(s) - sf(0) - f'(0)$
  - $\mathscr{L}[d^nf/dt^n] = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0)$

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Linearity Theorem

- Statement of Theorem
- $\mathscr{L}[af_1(t) + bf_2(t)] = a\mathscr{L}[f_1(t)] + b\mathscr{L}[f_2(t)]$
- Proof of linearity theorem
- $\mathscr{L}[af_1(t) + bf_2(t)] = \int_0^\infty e^{-st}[af_1(t)+bf_2(t)]dt$
  $= a\int_0^\infty e^{-st} f_1(t)dt + b\int_0^\infty e^{-st} f_2(t)dt = a\mathscr{L}[f_1(t)] + b\mathscr{L}[f_2(t)]$
- Importance: The Laplace transform of a ODE with constant coefficients is the sum of the transforms of all terms in the ODE

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First Shifting Theorem

- If $\mathscr{L}[f(t)] = F(s)$ then $\mathscr{L}[e^{at}f(t)] = F(s - a)$
- Get proof of first shifting theorem by modifying integral for F(s) to get F(s – a)
- $F(s) = \int_0^\infty e^{-st} f(t)dt$
- $F(s - a) = \int_0^\infty e^{-(s-a)t} f(t)dt =$
  $\int_0^\infty e^{-st}[e^{at}f(t)]dt = \mathscr{L}[e^{at}f(t)]$
- E.g. $\mathscr{L}[\sin(\omega t)] = F(s) = \omega^2/(s^2 - \omega^2)$ then
  $\mathscr{L}[e^{at}\sin(\omega t)] = F(s - a) = \omega^2/[(s - a)^2 - \omega^2]$

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Derivative Transformations

- Derivation for f' = df/dt
- $\mathscr{L}(f') = \int_0^\infty e^{-st} f'(t)dt = [e^{-st}f(t)]_0^\infty$
  $+s\int_0^\infty e^{-st} f(t)dt = [0 - f(0)] + s\mathscr{L}(f)$
- Derivative transforms of different orders
  - $\mathscr{L}[df/dt] = sF(s) - f(0)$
  - $\mathscr{L}[d^2f/dt^2] = s^2F(s) - sf(0) - f'(0)$
  - $\mathscr{L}[d^nf/dt^n] = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0)$

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Transform Notation

- Use lower case letters for dependent variables in original differential equation
- Time, t, is the usual independent variable
- If the ODE uses f(t), use F(s) for the notation of the transformed variable
  - Similarly use Y(s) for the transformed variable if the original variable is y(t)
  - Transform tables usually use f(t) and F(s)

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Solving Differential Equations

- Transform all terms in the differential equation to get an algebraic equation
  - For a differential equation in y(t) we get the transforms $Y(s) = \mathscr{L}[y(t)]$
  - Similar notation for other transformed functions in the equation $R(s) = \mathscr{L}[r(t)]$
- Solve the algebraic equation for Y(s)
- Obtain the inverse transform for Y(s) from tables to get y(t)
  - Manipulations often required to get from Y(s) equation to transforms in tables

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Differential Equation Example

- Transform all terms in the differential equation to get an algebraic equation
  - For $y'' + 3y' + 2y = 0$ we get the following: $s^2Y(s) - sy(0) - y'(0) + 3[sY(s) - y(0)] + 2Y(s) = 0$
- Solve the algebraic equation for Y(s)
  - $(s^2 + 3s + 2)Y(s) = (s + 3)y(0) + y'(0)$
  - $(s + 1)(s + 2) Y(s) = y(0) s + 3y(0) + y'(0)$

$$Y(s) = \frac{y(0)s}{(s+1)(s+2)} + \frac{3y(0)+y'(0)}{(s+1)(s+2)}$$

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Get Inverse Transform

- From transform table in Kreyszig (section 6.9, tenth edition, p. 249, entries 11 and 12 for a ≠ b)

$$F(s)=\frac{s}{(s-a)(s-b)} \Rightarrow f(t)=\frac{ae^{at}-be^{bt}}{a-b}$$

$$F(s)=\frac{1}{(s-a)(s-b)} \Rightarrow f(t)=\frac{e^{at}-e^{bt}}{a-b}$$

- We have

$$Y(s)=\frac{y(0)s}{(s+1)(s+2)}+\frac{3y(0)+y'(0)}{(s+1)(s+2)}$$

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Apply Inverse Transforms

$$Y(s)=\frac{y(0)s}{(s+1)(s+2)}+\frac{3y(0)+y'(0)}{(s+1)(s+2)}$$

$$y(t)=y(0)\frac{(-1)e^{1t}-(-2)e^{-2t}}{-1-(-2)}+[3y(0)+y'(0)]\frac{e^{1t}-e^{-2t}}{-1-(-2)}$$

$$y(t)=y(0)\left[2e^{-2t}-e^{-t}\right]+\left[3y(0)+y'(0)\right]\left[e^{-t}-e^{-2t}\right]$$

- Solution valid for any initial conditions

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About Inverse Transformations

- Use transform table
- May be able to use the first shifting theorem discussed earlier
- Method of partial fractions will be discussed later in this lecture
- Use second shifting theorem
  - Second shifting theorem uses definitions of Heavyside unit function and Dirac delta function discussed next lecture

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Nonhomogenous ODE Example

- Look at nonhomogeneous case for same ODE analyzed previously
- y'' + 3y' + 2y = r(t)
- Transform all terms in the equation and rearrange to solve for Y(s)
  - $s^2Y(s) - sy(0) - y'(0) + 3[sY(s) - y(0)] + 2Y(s) = R(s) = \mathcal{L}[r(t)]$
  - $(s^2 + 3s + 2)Y(s) = (s + 3)y(0) + y'(0) + R(s)$
  - $(s + 1)(s + 2)\,Y(s) = y(0)\,s + R(s) + 3y(0) + y'(0)$

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Nonhomogenous Example II

- Solve (s + 1)(s + 2) Y(s) = y(0) s + R(s) + 3y(0) + y'(0) for Y(s)

$$Y(s)=\frac{R(s)}{(s+1)(s+2)}+\frac{y(0)s+3y(0)+y'(0)}{(s+1)(s+2)}$$

$$Q(s)\equiv\frac{1}{(s+1)(s+2)}$$

$$Y(s)=R(s)Q(s)+\left[y(0)s+3y(0)+y'(0)\right]Q(s)$$

- Q(s), (H(s) or W(s) also used for Q) is called the system or transfer function

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Nonhomogenous Example III

- Second term same as homogenous case

$$Y(s)=\frac{R(s)}{(s+1)(s+2)}+\frac{y(0)s+3y(0)+y'(0)}{(s+1)(s+2)}$$

$$\frac{y(0)s+3y(0)+y'(0)}{(s+1)(s+2)}=\frac{y(0)s}{(s+1)(s+2)}+\frac{3y(0)+y'(0)}{(s+1)(s+2)}$$

- Kreyszig, Table 6.9, No. 11 & 12

$$\mathfrak{I}^{-1}\left[\frac{s}{(s+1)(s+2)}\right]=\frac{-1e^{-1t}-(-2)e^{-2t}}{-1-(-2)}$$

$$\mathfrak{I}^{-1}\left[\frac{1}{(s+1)(s+2)}\right]=\frac{e^{-1t}-e^{-2t}}{-1-(-2)}$$

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Nonhomogenous Example IV

- Pick some r(t) and get corresponding R(s) to get example solution
  - Pick r(t) = sin ωt so R(s) = $\omega/(s^2 + \omega^2)$

$$Y(s) = \frac{y(0)s + R(s)}{(s+1)(s+2)} + \frac{3y(0) + y'(0)}{(s+1)(s+2)} = \frac{y(0)s}{(s+1)(s+2)}$$

$$+ \frac{\omega}{(s+1)(s+2)(s^2+\omega^2)} + \frac{3y(0)+y'(0)}{(s+1)(s+2)}$$

- Cannot find $\omega/(s+a)(s+b)(s^2 + \omega^2)$ in transform table

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Partial Fractions

- Method to convert fraction with several factors in denominator into sum of individual factors (in denominator)
- Example is 1/(s+a)(s+b)
- Write 1/(s+a)(s+b) = A/(s+a) + B/(s+b)
- Multiply by (s+a)(s+b) and equate coefficients of like powers of s
  - 1 = A(s + b) + B(s + a)
  - A + B = 0 for s^1 terms and 1 = bA + aB for s^0 terms and A + B = 0 for s^1 terms

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Partial Fractions II

- A + B = 0 for s^1 terms and 1 = bA + aB for s^0 terms
- Solving for A and B gives A = -B = 1/(b – a)

- Result: 1/(s+a)(s+b) = 1/[(b – a)(s + a)] – 1/[(b – a)(s + b)]
- This actually matches a table entry
- Follow same basic process for more complex fractions
- Special rules for repeated factors and complex factors

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Partial Fraction Rules

- Repeated fractions for repeated factors

$$\frac{1}{\cdots(s+a)^n\cdots} = \cdots + \frac{A_n}{(s+a)^n} + \frac{A_{n-1}}{(s+a)^{n-1}} + \cdots + \frac{A_2}{(s+a)^2} + \frac{A_1}{s+a} + \cdots$$

- Complex factors (s + α - iβ)(s + α + iβ)

$$\frac{1}{\cdots(s+\alpha-i\beta)(s+\alpha+i\beta)\cdots} = \cdots + \frac{As+B}{(s+\alpha)^2+\beta^2} + \cdots$$

- Pure imaginary factor is complex factor with α = 0

$$\cdots + \frac{1}{s^2+\beta^2} + \cdots = \cdots \frac{1}{(s-i\beta)(s+i\beta)} \cdots = \cdots + \frac{As+B}{s^2+\beta^2} + \cdots$$

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Example: Apply Partial Fractions

- Split expression not found in table into components (complex factor, $s^2 + \omega^2$)

$$\frac{\omega}{(s+1)(s+2)(s^2+\omega^2)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{Cs+D}{(s^2+\omega^2)}$$

- Multiply by original denominator

$$\omega = A(s+2)(s^2+\omega^2) + B(s+1)(s^2+\omega^2) + (Cs+D)(s+1)(s+2)$$

$$\omega = A(s^3+2s^2+\omega^2 s+2\omega^2) + B(s^3+s^2+\omega^2 s+\omega^2)$$
$$+ C(s^3+3s^2+2s) + D(s^2+3s+2)$$

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Continuing Example

- Equate coefficients of like s powers

$$\omega = A(s^3+2s^2+\omega^2 s+2\omega^2) + B(s^3+s^2+\omega^2 s+\omega^2)$$
$$+ C(s^3+3s^2+2s) + D(s^2+3s+2)$$

- 0 = A + B + C (s^3 terms)
- 0 = 2A + B + 3C + D (s^2 terms)
- 0 = Aω² + Bω² + 2C + 3D (s^1 terms)
- ω = 2Aω² + Bω² + 2D (s^0 terms)

- Solve for A, B, C, and D

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Solve for A, B, C, and D

- Gauss elimination for augmented matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 3 & 1 & 0 \\ \omega^2 & \omega^2 & 2 & 3 & 0 \\ 2\omega^2 & \omega^2 & 0 & 2 & \omega \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 2-\omega^2 & 3 & 0 \\ 0 & -\omega^2 & -2\omega^2 & 2 & \omega \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 2-\omega^2 & 3 & 0 \\ 0 & 0 & -3\omega^2 & 2-\omega^2 & \omega \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 2-\omega^2 & 3 & 0 \\ 0 & 0 & 0 & 2-\omega^2-\dfrac{9\omega^2}{2-\omega^2} & \omega \end{bmatrix}$$

Backsolve for A, B, C, and D

$$D = \frac{\omega}{2-\omega^2-\dfrac{9\omega^2}{2-\omega^2}} = \frac{\omega(2-\omega^2)}{(2-\omega^2)^2-9\omega^2} \qquad \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 2-\omega^2 & 3 & 0 \\ 0 & 0 & 0 & 2-\omega^2-\dfrac{9\omega^2}{2-\omega^2} & \omega \end{bmatrix}$$

$$C = \frac{-3D}{2-\omega^2} = \frac{-3}{2-\omega^2}\frac{\omega(2-\omega^2)}{(2-\omega^2)^2-9\omega^2} = \frac{-3\omega}{\left(2-\omega^2\right)^2-9\omega^2}$$

$$B = C + D = \frac{-3\omega}{(2-\omega^2)^2-9\omega^2} + \frac{\omega(2-\omega^2)}{(2-\omega^2)^2-9\omega^2} = \frac{-\omega-\omega^3}{(2-\omega^2)^2-9\omega^2}$$

$$A = -B - C = -\frac{-\omega-\omega^3}{(2-\omega^2)^2-9\omega^2} - \frac{-3\omega}{(2-\omega^2)^2-9\omega^2} = \frac{4\omega+\omega^3}{(2-\omega^2)^2-9\omega^2}$$

Write Y(s) – Get Transforms

$$Y(s) = \frac{y(0)s}{(s+1)(s+2)} + \frac{\omega}{(s+1)(s+2)(s^2+\omega^2)} + \frac{3y(0)+y'(0)}{(s+1)(s+2)}$$

$$\frac{\omega}{(s+1)(s+2)(s^2+\omega^2)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{Cs+D}{(s^2+\omega^2)} = \frac{\dfrac{4\omega+\omega^3}{(2-\omega^2)^2-9\omega^2}}{s+1} + \frac{\dfrac{-\omega-\omega^3}{(2-\omega^2)^2-9\omega^2}}{s+2} + \frac{\dfrac{-3\omega}{\left(2-\omega^2\right)^2-9\omega^2}s + \dfrac{\omega(2-\omega^2)}{(2-\omega^2)^2-9\omega^2}}{(s^2+\omega^2)}$$

Write Y(s) – Get Transforms II

$$Y(s) = \frac{y(0)s}{(s+1)(s+2)} + \frac{\dfrac{4\omega+\omega^3}{(2-\omega^2)^2-9\omega^2}}{s+1} + \frac{\dfrac{-\omega-\omega^3}{(2-\omega^2)^2-9\omega^2}}{s+2} - \frac{3\omega}{\left(2-\omega^2\right)^2-9\omega^2}\frac{s+\dfrac{(2-\omega^2)}{3}}{(s^2+\omega^2)} + \frac{3y(0)+y'(0)}{(s+1)(s+2)}$$

$$y(t) = y(0)\frac{-2e^{-2t}-(-1)e^{-t}}{-2-(-1)} + \frac{\left(4\omega+\omega^3\right)e^{-t}}{(2-\omega^2)^2-9\omega^2} - \frac{\left(\omega+\omega^3\right)e^{-2t}}{(2-\omega^2)^2-9\omega^2} - \frac{3\omega}{(2-\omega^2)^2-9\omega^2}\left[\cos\omega t + \frac{2-\omega^2}{3}\frac{\sin\omega t}{\omega}\right] + \left(3y(0)+y'(0)\right)\frac{e^{-2t}-e^{-t}}{-2-(-1)}$$

Laplace Transform Solution

$$y(t) = y(0)\left(2e^{-2t}-e^{-t}\right) + \frac{\left(4\omega+\omega^3\right)e^{-t}}{(2-\omega^2)^2-9\omega^2} - \frac{\left(\omega+\omega^3\right)e^{-2t}}{(2-\omega^2)^2-9\omega^2} - \frac{3\omega\sin\omega t}{(2-\omega^2)^2-9\omega^2} + \frac{\omega(2-\omega^2)\cos\omega t}{(2-\omega^2)^2-9\omega^2} - \left(3y(0)+y'(0)\right)\left(e^{-2t}-e^{-t}\right)$$

- Usual solution starts with numerical values of ω and initial conditions
  - These numerical values simplify the result
  - Solution presented here is valid for any ω and any initial conditions

Other Applications

- We can apply this to a system of equations for $y_i(t)$
  - Transform all equations from $y_i(t)$ to $Y_i(s)$
  - Solve simultaneous algebraic equations for each $Y_i(s)$
  - Get inverse transforms for $y_i(t)$
  - Will consider systems of equations in the next class period